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When Do Gifted High School Students Use Geometry to Solve Geometry Problems?

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This article describes the following phenomenon: Gifted high school students trained in solving Olympiad-style mathematics problems experienced conflict between their conceptions of *effectiveness* and *eleganc*e (the EEC). This phenomenon was observed while analyzing clinical task-based interviews that were conducted with three members of the Israeli team participating in the International Mathematics Olympiad. We illustrate how the conflict between the students' conceptions of effectiveness and elegance is reflected in their geometrical problem solving, and analyze didactical and epistemological roots of the phenomenon.

Prologue

Saul[1], a 16-year-old member of the team representing Israel in the International Mathematics Olympiad (IMO), was interviewed in a study of gifted and average students' strategic behaviors in mathematical problem solving[2]. At the beginning of the interview, the following dialogue took place between Saul and the interviewer:

Interviewer: How do you approach difficult geometry problems?

Saul: Generally speaking, I try to understand what the fuzziest point in the problem is and then I apply my intuition to this point. . . . If I don't have any idea what to do, I just use different, not nice, methods.

Interviewer: What do you mean?

Saul: I use the special methods that I have learned, like complex numbers in geometry, or trigo . . . where there is no choice.

Interviewer: Why do you think that these methods are not nice?

Saul: They could be nice, but . . . [sighs, pauses 5 seconds].

Interviewer: Which methods are "nice"?

Saul: Nice is when I have a geometry problem and I solve it by means of classic geometry.

In this dialogue, Saul expressed the opinion that some algebra-laden methods in geometry problem solving may be effective but are not nice. Furthermore, we observed that Saul and some other top-achieving Olympians experience similar mixed feelings when solving problems presumably solvable within Euclidian geometry, using advanced analytical or trigonometric techniques. As researchers, we became intrigued by apparent conflict between the gifted students' conceptions of *effectiveness* and *elegance* in problem solving. Hereafter, we will refer to this conflict as the EEC.

In this paper, the EEC is explored as a force potentially guiding problem-solving behaviors of mathematically

gifted students. Specifically, we analyzed geometry problem-solving experiences of three top-achieving Olympians and approach the following questions:

1. How is the EEC reflected in the top-achieving Olympians' geometry problem solving during clinical task-based interviews?
2. What are possible didactical and epistemological roots of the EEC?

Theoretical Background

In this section, we briefly discuss the concepts of effectiveness and elegance and review the relevant literature concerning problem solving and intellectual giftedness.

Effectiveness and Elegance in Mathematical Problem Solving

A problem-solving method is often considered effective if it leads to a solution of a given problem without unnecessary effort, and elegant if it is characterized by clarity, simplicity, parsimony, and ingenuity (Baker, 2004; Dreyfus & Eisenberg, 1986; Krutetskii, 1976; Silver & Metzger, 1989). Note that this assertion, produced as an extract from the aforementioned literature sources, has (at best) some communicative value, but cannot serve as a satisfactory definition of the concepts of effectiveness and elegance. Indeed, in the above assertion, effectiveness and elegance appear as different, yet not completely alien concepts. Moreover, the explanatory terms *clarity*, *simplicity*, *parsimony*, and *ingenuity* are not better institutionalized than the target term elegance (see Baker, 2004, and Sinclair, 2004, for philosophical and educational views on this issue, respectively). However, this sort of criticism is not a barrier for doing research on students' conceptions of effectiveness and elegancy. The entire history of psychological and educational research teaches us that exploration of complicated sociocultural concepts in absence of their institutionalized definitions is both possible and desirable. To study effectiveness and elegance, we consider these concepts as one's multiply encoded cognitive/affective configurations, to which the holder attributes some kind of truth value. As such, effectiveness and elegancy, specified (but not defined) above, can be seen ultimately as matters of one's belief system (Goldin, 2002).

Beliefs of several distinguished mathematicians of the first half of the 20th century about aesthetics in problem solving essentially influenced the state of the art. For instance, Hadamard (1945) and Poincaré (1946) acknowledged the role of effectiveness and elegance in mathematical inquiry, but they did not fully concur regarding a relationship between these two concepts. On the one hand, Hadamard argued that sense of elegance enables a mathematician to make unconscious choices about what direction to take during problem solving. Commenting on Hadamard, Silver and Metzger (1989) noticed that his view might suggest that any effective choice in problem solving is elegant. On the other hand, Poincaré suggested that effective choices are not necessarily the elegant ones; he also pointed out that considerations of elegance can either be fruitful or lead away from effective solutions.

Educational research provides us with some evidence that considerations of elegance are inherent in problem solving of the intellectually gifted. Krutetskii (1976) observed that mathematically capable children are typically striving "for the cleanest, simplest, shortest and thus most 'elegant' path to the goal" (p. 284), whereas average students pay little attention to aesthetics of their solutions. Silver and Metzger (1989) have shown that professional mathematicians, as well as mathematics graduate students, manifest aesthetic considerations while solving problems or evaluating solutions by others. They found that the emotional side of evaluation of the solution's elegance can be either positive or negative. For instance, one of the participants in their study, a mathematics professor, negatively evaluated algebraic approaches to geometry problems. Silver and Metzger also suggested that aesthetics appears to serve as a basis for linking metacognitive processes, such as planning and monitoring, and thus can fit into Schoenfeld's (1985) model of problem solving. They called for an investigation of the role of aesthetics and different kinds of parsimony in mathematical problem solving. This call was manifested, with some variations, by many authors during the last 15 years (e.g., Dreyfus & Eisenberg, 1986, 1996; McGregor, 2001; Sinclair, 2001, 2004).

Recently, Sinclair (2004) distinguished three roles of aesthetics in doing mathematics: (a) the evaluative, which involves ad hoc judgment about beauty and elegance of proofs and solutions; (b) the generative, which involves nonpropositional modes of reasoning in problem solving, as those described by Hadamard (1945) and Poincaré (1946); and (c) the motivational, which involves cognitive/affective processes, in which one's mathematical preferences and taste are developed. According to Sinclair (2004), the first role of the aesthetics is better documented than the second and third ones. We also learn from her paper that, with rare exceptions (e.g., Krutetskii, 1976), the conception of elegance in problem solving is explored within the elite community of professional mathematicians and mathematics graduate students. Little is known about

the role of aesthetics in problem solving of gifted adolescents, and, in particular, of top-achieving Olympians.

Five-Factor Model of Intellectual Giftedness

In the forthcoming discussion of the EEC, we will utilize the five-component framework of intellectual giftedness. Sternberg and Davidson (1983) have identified three major cognitive components that are responsible for the solutions of insight problems by the gifted: encoding, combination, and comparison. Gorodetsky and Klavir (2003) suggested extending the model and incorporating in it two additional components—retrieval and goal directness.

In short,

- *Encoding* refers to the subprocess by which the solver extracts information from a given problem. The gifted were found to encode deep-structure relations of the problem and to ignore irrelevant features embedded in the problem.
- *Combination* occurs when a problem solver combines encoded information, its semantic interpretation, and evoked procedural knowledge into a solution structure. The gifted were found to be highly selective while combining the encoded information. (A combination was considered selective if it combined pieces of information into an integrative solution process.)
- *Comparison* alludes to the solver's search for a pattern that may lead to a solution, and concurrent comparison of that pattern with possible solution structures attained in past learning. The gifted were found to see the equivalence of two problems at their deep structure level when the problems were different on the surface.
- *Retrieval* refers to the activation of concepts and terms that enable the interpretation of a given problem in the problem solver's terms. It also refers to retrieval of procedural knowledge. The gifted seem to use these two kinds of retrieval almost automatically.
- *Goal directness* captures the fact that gifted are often faster than average problem solvers in reaching solutions. The gifted also demonstrate reflection in action, that is, continuous self-regulating and tuning during problem solving.

Gorodetsky and Klavir (2003) utilized the above five-component model in a nonmathematical context: In their study, gifted and average middle school students were given insight verbal puzzles and then answered written questionnaires in which they reflected on the solution processes. In our study, top-achieving high school Olympians were given insight geometry problems in a clinical task-based interview setting. Despite these methodological differences, we found the five-component model of the intellectual giftedness a useful framework for the forthcoming discussion of the students' conceptions of effectiveness and elegance in problem solving. Specifically, we will interpret the students solutions in terms of the above five subprocesses.

Method

Participants

Mike (age 17), Alex (age 16), and Saul (age 16)—the students who volunteered to take part in the study—were invited to the Israeli Mathematical Competitions at least 2 years prior to the interviews. This national project focuses on developing mathematical giftedness and preparing for the IMO. In this project, the students were intensively trained in advanced problem-solving methods and participated in various kinds of mathematical competitions, including international ones. There is broad consensus that high-achieving Olympians can be seen as intellectually gifted (e.g., Berzsenyi, 1999; Grassl & Mingus, 1999; Nokelainen, Tirri, & Campbell, 2004). We deemed these students gifted based on their distinguished records: They were awarded with gold and silver medals of several IMOs and, while in high school, gained credit for the Bachelor of Science degree in mathematics.

Interview Procedure

The participants were interviewed individually for 2 to 3 hours. Each interview was videotaped using a camera trained on the interviewee. The interviewer (the first author) was situated aside so that he could observe the students' writing. At the beginning of each interview, the interviewer thanked the student for agreeing to participate in the interview, which was not a part of the process of preparation/selection to the IMO. He informed the participants that the purpose of the interview was to learn about their ways of thinking while solving various mathematical problems, and that the interview was a part of the research project aimed at development problem-solving skills of regular high school students (Koichu, 2003; Koichu, Berman, & Moore, 2004).

To warm up, the participants were asked to explain how they usually approach difficult Olympiad problems (e.g., the dialogue quoted in the prologue occurred at this stage of the interview). Then the students were given five problems and instructed to think out loud while solving the problems. This part of interview was administered in accordance with Erickson and Simon's (1993) recommen-

dations. Namely, if the student remained silent for more than 15 seconds, the interviewer prompted him in a neutral manner (e.g., "Keep talking" and "Don't be silent"). The interviewer also could occasionally ask clarification questions (e.g., "What did you do?" and "You are building the perpendicular, right?"). The amount of time devoted to each problem was up to the interviewee. When the student decided to stop working on a particular problem, he addressed retrospective questions concerning the given problem (e.g., "Please explain how you solved the problem," "Tell me how you got the idea," and "How difficult or new was the problem?"). If and when the interviewee evaluated his problem-solving approach as unsatisfactory (e.g., by indicating that it was "wrong" or "not nice") he was prompted to try the problem again.

Interview Problems

During the interview, the students were given two geometry problems, two algebra problems, and one open task for investigation. In this paper we focus on the geometric part of the interviews. The two geometry problems (see Figure 1) were chosen as potentially challenging but feasible for the participants. Solid geometrical background and well-polished problem-solving skills were needed to handle each problem. Both problems can be solved either by means of Euclidian geometry or using trigonometry.

The Angle Problem, taken from Prasolov (1995), is difficult at a first glance because the givens are not assembled in one construction. We expected that the interviewees were not familiar with this problem. The Bisector Problem is the famous Steiner-Lehmus Theorem, which looks like an easy-to-prove statement but, in fact, is not. The discussion of this theorem may be found in Coxeter and Greitzer (1967) or in many Internet mathematical forums. It was expected that the interviewees might recognize the Steiner-Lehmus Theorem in the Bisector Problem because it was included in their early preparatory course for mathematical competitions as a part of geometry classics. This problem was given in the interview to see whether the students would try to remember the solution or to attack the problem as a new one.

Analysis

The interview data consisted of videotapes, transcripts of the videotapes, the students' written work, and the experimenter's field notes. The data were analyzed using "the open interpretation of large episodes by an individual analyst" (Clement, 2000, p. 548). Clement argued that this approach is useful for constructing initial explanatory models of cognitive processes, inferred from the naturalistic observations. Specifically, suggestions about the role of the students' conceptions of effectiveness and elegancy in problem solving were formulated by abduction. Abduction is referred to as the process of producing a model that, if true, would account for the phenomenon in question (Pierce, 1958, as cited in Clement). Thus, the concern about viability rather than validity of the outcomes of the analyses is relevant. As von Glasersfeld and Steffe (1991) pointed out, "The most one can hope for is that the model fits whatever observations one has made and, more important, that it remains viable in the face of new observations" (p. 98).

Angle Problem: Let ABC be an acute angle triangle, AH is the longest altitude of the triangle and BM is a median, AH = BM. Prove that $\angle B \leq 60°$.

Bisector Problem: Prove the following statement:

If two bisectors of a triangle are equal, then it is an isosceles triangle.

Figure 1. The interview problems

Limitations

Limitations of the presented research stem chiefly from:

- the small number of observations, which raises a concern for viability, as indicated above;
- the nature of the thinking-aloud procedure, which, according to Silver and Metzger (1989), might suppress the "natural" problem-solving behaviors of the participants; and
- the nature of the open-ended analysis, which cannot be free from bias.

These limitations prescribe modesty in generalization of the research findings. However, we point out that the research questions (see prologue) are within the reach of the described methodology.

Findings and Discussion

Six interview episodes are presented in this section at different levels of detail. In accordance with the applied methodology, the findings are communicated in a format of analyst-written stories with short excerpts from the transcripts.

Episode 1: Mike Solves the Angle Problem

During the first reading of the Angle Problem, Mike drew triangle ABC with median BM and altitude AH (see Figure 2, which keeps the proportions and notations of the original Mike's drawing, except for italicized letter *D* added for the sake of coherence).

Then, after a 15-second pause:

Mike: Let's try to build here a parallel line [segment MK parallel to AH]. KM is a half of AH . . . What is given? Aha . . . they are equal [AH = BM]. We should use it since there is nothing more to use. . . . No, it is also given that the angles are acute [ABC is an acute angle triangle], but it is usually given to avoid a situation when they [altitudes] may be outside. . . Oh, it is also given that AH is the largest . . . This means that BC is the smallest . . .

Interviewer: Why?

Mike: Because of the areas. The areas are constant . . . CB times AH equals AB times this [points to CD] . . . Now. . . . BC is the smallest side. To make B less than 60°, we want to prove that AB is the largest [side]. It implies . . . We use what we want to prove, but . . . this is right, even though I don't see now how to prove it . . . OK, let's attack this angle [angle B]. $\angle B = 90° - \angle HAB$. We need to prove that $\angle HAB > 30°$. How to use that this median is equal to this altitude [the given BM = AH]. Aha, to prove that $\angle HAB > 30°$, means to prove that in the right-angle triangle [triangle BAH] $2 \cdot HB > AB$, it is enough . . . [15 second pause] Anyway, we should use the given . . . AH = BM.

The processes of encoding and comparison are noticeable in the above episode. Indeed, Mike first built parallel lines and only then considered carefully what was given in the problem. Apparently, something in the problem's formulation activated the parallel lines scheme that Mike had used successfully in the past. At this stage, Mike did not know whether his auxiliary construction would work. He left it for a while and approached the problem from end to beginning trying to substitute the problem's goal with an equivalent and more manageable one. For this reason, we indicate a highly selective combination process. The phrase

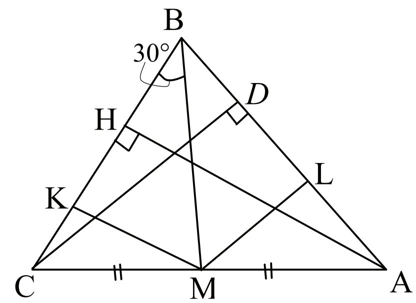

Figure 2. Mike solves the Angle Problem

"We use what we want to prove, but . . . this is right, even though I don't see now how to prove it" enlightens the nature of Mike's heuristic search and also tells us that he relied on his empirical judgment. At this stage, the main difficulty for Mike was how to use the given AH = BM, and he found himself captured in a circle.

Mike's next assertion is prominent:

Mike: I would like to know how difficult the problem is. If it is difficult, I can represent everything using the formulas, and at the end it will work, I just want to save the effort . . . [Pause 10 seconds, the interviewer keeps silence]

When the first attack on the problem did not lead to a quick solution, Mike asked for a very specific assistance—the information about the difficulty of the problem. Mike knew what kind of problems can be expected at different mathematical competitions and preparatory sessions but did not know what may be asked in the interview. Apparently, depending on the expected level of difficulty of the problem, Mike retrieved different problem-solving methods. During the 10-second pause, he decided that there is no need for algebra because he made an important observation about angle MBL, and completed the solution very quickly, as follows:

Mike: Aha, this segment is two times this one [writing: 2MK = BM]. From here, $\angle MBL = 30°$ [Mike denotes 30° on the drawing]. OK, now, why is the second angle [angle ABM] less than or equals to 30°? Because . . . [Pause 10 seconds, drawing ML] because ML < MK, that's why $\angle MBL \leq 30°$, everything is less than 60° [$\angle B \leq 60°$].

Prominent goal-directedness is also indicated in the above fragment. In the retrospective part of the interview, Mike explained:

Mike: [I first built parallel lines] since this is what usually helps. In such situations when something is divided into two parts, as with a median, one has to build parallel lines. . . . There are not many lines and not many choices to add something. I did what I used to do.

Episode 1 Discussion

In the above episode, Mike acted in remarkable accordance with the model of giftedness outlined earlier in this paper. The interplay of the five subprocesses by Gorodetsky and Klavir (2003) are easily seen in his behavior, as specified above. We observed that Mike was ready to implement the powerful but effort-demanding algebraic approach when he did not see how the geometry-laden problem-solving schemes (i.e., parallel lines and area schemes) could work. From Mike's standpoint, it was fortunate that the algebraic approach was not needed. He did not have to "use formulas" because the problem turned out not to be too difficult. Thus, considerations of elegance played a generative role in Mike's reasoning, to use Sinclair's (2004) terms.

Episode 2: Mike Solves the Bisector Problem

After reading the Bisector Problem, Mike was silent for 15 seconds.

Interviewer: What are you thinking about?

Mike: I am choosing the way [to solve the problem]. On the one hand, the problem can be solved geometrically, but I am not sure how . . . On the other hand, I am sure that it can be solved algebraically, but I feel too lazy to do so. Indeed, I can write a, b, c [to denote the sides of a triangle] to compute everything and at the end, I know exactly, for 100%, everything will be all right. [Six second pause. Mike looks at the interviewer, apparently expecting the interviewer to stop him, but the interviewer does not]. Okay, I can do it.

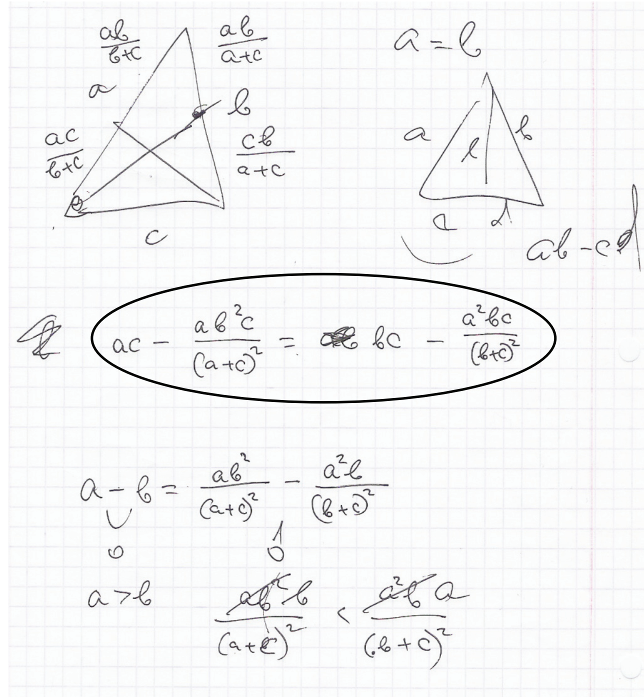

Figure 3. Mike solves the Bisector Problem algebraically

Mike retrieved the formulas for the parts in which a bisector's endpoint divides the corresponding side of a triangle (see notations at the left drawing on Figure 3).

After finishing, he explained that these formulas belong to the "standard" set of methods and that he knows how to prove them either geometrically or algebraically. Based on the given condition that the bisectors are equal in the above formulas, Mike wrote the equation (framed with an ellipse in Figure 3), but almost did not try to manipulate it. Instead, he analyzed the equation qualitatively and showed that the assumption "a > b" leads to contradiction: the left side is positive, and the right side of the equation is negative. As the last step, Mike referred to symmetry considerations in order to conclude that the assumption "a < b" would also lead to contradiction, and thus a = b. This multistep solution was done very quickly, in about 5 minutes.

Interviewer: What was the key moment in your thinking?

Mike: When I decided to solve the problem algebraically. I was almost sure that I would succeed.

Mike chose not to take chances and to solve the problem algebraically. When he made this decision, the rest was determined to be easy for him. He solved the problem by

translating its geometrical content into an algebraic equation. Beyond his impressive mathematical background, Mike demonstrated a remarkable goal-directedness and reflection in action: He came back to the geometrical meaning of the algebraic equation and solved the problem effectively and, to us, elegantly.

Nevertheless, Mike showed that he was not happy with his solution, and the interviewer prompted him to solve the problem geometrically. The student tried to do so, but realized in about 5 minutes that he was distracted by ideas he had used in the algebraic proof. He said:

Mike: The first solution intersects my thinking, I am just trying to translate algebra into geometry. It is not fair. . . . I am sure that there is a purely geometrical solution, but it is more difficult to find.

Interviewer: Why?

Mike: It is unclear what to do. I mean . . . algebra. . . . For example, we should prove something—Okay, we represent it algebraically, and at the end everything works.

Interviewer: And what exactly is difficult in geometrical solutions? Sometimes there are very short . . .

Mike: One should have an insight, to figure out what to do. Sometimes, if you have experience. . . . You can get it, but very often you cannot . . . or it takes a lot of time.

Episode 2 Discussion

Trained to solve problems at mathematical competitions under pressure and time constraints, Mike chose the most effective (the least time-consuming) solution method. Dreyfus and Eisenberg (1986) noted that an opportunity to have an immediate picture of a solution overrides aesthetic concerns in expert problem solving. Mike's problem-solving behavior matches this observation, especially in the second episode. However, a closer look at the above two episodes elicits the more complicated picture of Mike's reasoning. We think that the student behaved consistently in spite of the fact that he handled the first problem geometrically and the second one algebraically.

The heuristics driving Mike to choose between geometry-laden and algebra-laden methods in the two episodes is, in effect, a paraphrase of the well-known Occam's principle of parsimony (e.g., Baker, 2004): One should not make more efforts than the minimum needed. Mike just takes into account different kinds of intellectual efforts. Indeed, he chose problem-solving approaches based on the estimation of effort needed either to get an insight, as in geometry-laden solutions, or to perform a not insightful routine, as in utilizing formulas aimed at composing equation. However, these are not the only heuristics, especially when time is not restricted, as in the described interview. Mike expressed more satisfaction when solving the problems geometrically than algebraically; the EEC was indicated when Mike produced the algebraic solution. It seems that Mike likes overcoming intellectual challenges; inventing a pure geometric solution presents a greater challenge to him than solving the problem by algebraic means. This observation is consistent with Harel's (1998) premise that the need to be puzzled is among the basic intellectual needs of a human being.

Our next comment is about the attempt to invent a geometry solution after providing the algebraic one. Mike could not just forget the previous method. Here, one can see the operation of the Law of Non-Deletion: "'Old ideas remain, although the system may refer to them less and less frequently. Representations are not actually deleted from memory" (Davis, 1984, p. 109). This implies that the order in which a problem solver tries different approaches may be of great importance. It seems that a lot of confidence and developed sense of elegance are needed in order to start solving a problem by looking for an aesthetically pleasing idea, whereas a less demanding routine method is likely to lead to an effective solution.

Episode 3: Alex Solves the Angle Problem

When Alex read the problem, he immediately built triangle ABC, altitude AH, median AM, and without any pause doubled the median and constructed parallelogram $ABCB_1$ (see Figure 4).

Interviewer: Why did you construct a parallelogram?

Alex: I think it may lead to something, since it helps in many problems.

Like Mike, Alex started from an auxiliary construction that helped him in the past. The build-a-parallelogram scheme was retrieved almost automatically because a median of the triangle was among the given information. Then, Alex looked at the drawing for 10 seconds in silence, but, unlike Mike, he did not quickly figure out how to use the auxiliary construction to solve the problem, and started a new attempt:

Alex: It would be better to compute everything by formulas.

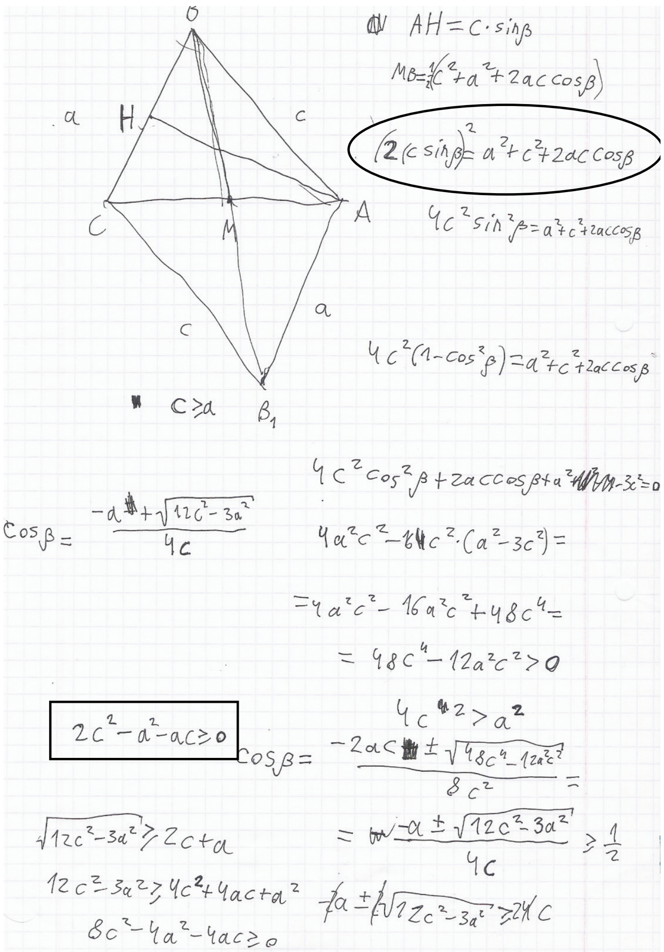

Figure 4. Alex solves the Angle Problem

He then denoted the sides of parallelogram $ABCB_1$, *a* and *c*. He then represented the given condition AH = BM as an equation (framed with an ellipse in Figure 4), using the theorem of cosines for triangle BCB_1 and the definition of sine of the target angle B [triangle ABH]. Alex transformed the equation into a quadratic one regarding cos *B*, expressed cos *B* by the formula for roots of a quadratic equation and formulated the new target: to prove that $\cos B \geq \frac{1}{2}$. At that stage, the student said: "Unfortunately, it may all be useless." Nevertheless, Alex continued the solution. Finally, he obtained the inequality matching the target one, which holds under the condition $c \geq a$ [this inequality is framed with a rectangle in Figure 4]. The condition $c \geq a$ was derived by Alex orally from the given that AH is the largest altitude, and thus, he solved the problem.

Alex: I've finished. It is not a difficult problem.

Note that Alex's and Mike's ideas about what constitutes a difficult problem were markedly different. Mike said that the Angle Problem was not difficult because he solved it geometrically and was not compelled to use algebraic approach. Alex said that the problem was not difficult because he solved it relatively quickly (in 7 minutes).

Interviewer: Anyway, you wrote a lot, but there is a very short geometrical solution . . .

Alex: Maybe, but in order to find a geometrical solution you usually spend much more time. Personally, I like geometrical solutions, but I look for them only if I cannot solve a problem by trigonometry.

After this dialogue, Alex said that he wanted to solve the problem geometrically. He found the solution in 25 minutes as a result of four different attempts. Alex tried various auxiliary constructions aimed at combining the given AH = BM in one triangle. The typical sentiment behind several of Alex's comments was "It is not clear yet where it may lead, but let's try." It seemed as if Alex mentally scanned a set of methods available to him (e.g., symmetry, perpendiculars, rotation), realizing a sophisticated kind of trial and error strategy. The processes of encoding, comparison, combination, and retrieval noticeably intertwined during these 25 minutes, but the considerations of elegance were invisible. Ultimately, he succeeded when using his first idea: namely, doubling the median and building a parallelogram, and looked satisfied by this solution. Unlike Mike, Alex was not captured by the ideas of his first solution. He explained it as follows: "Algebraic solution . . . there are no ideas in it. The only idea is to use the formulas."

Episode 3 Discussion

As an experienced Olympian, Alex was looking for the most effective, or the least time-consuming, problem-solving method, and considerations of elegance were only complementary to him. The EEC was reflected in an ad hoc negative assertion regarding noninsightful algebraic routine, and thus, played the evaluative role. It is unclear whether or not the EEC played a generative role in the student's online reasoning.

Episode 4: Alex Refuses to Solve the Bisector Problem

Alex read the problem, drew an appropriate picture and said:

Alex: I have seen this problem before. This is one of the Olympiad problems that any-

	body knows. I don't remember its solution exactly, but I remember the idea.
Interviewer:	Algebraic or geometric?
Alex:	Geometric. The idea was to build parallel lines to the bisectors.

Afterwards, Alex tried (unsuccessfully) to remember the solution he had been taught and did not try to think of his own ideas. He decided to stop working on the Bisector Problem because it was "not interesting."

Episode 4 Discussion

Alex did not accept the intellectual challenge. He was not puzzled by a problem whose solution may be found in a textbook. This behavior may also be explained by the principle of parsimony; the aesthetic behaviors were not observed in this episode. Straightforward memory retrieval of problem-solving schemata seems to be more important to Alex than to his teammates. This will be evident from comparison of Episodes 2, 4, and 6.

Episode 5: Saul Solves the Angle Problem

Saul solved the Angle Problem in about 10 minutes. While reading the problem for the first time, Saul drew triangle ABC, median BM, and altitude AH, as seen in Figure 5 (Figure 5 keeps the proportions and the notations of Saul's original drawing, except for the italicized letter *D*, which we added for the sake of coherence).

The student started by exploring the givens:

Saul:	It is given that AH is the greatest altitude, then . . . BC is the smallest side, since the area of triangle . . . area . . . I should somehow use the fact that AH = BM . . .

In 2 minutes, he figured out how to use this given. Saul drew the altitudes of triangles ABM and CBM—they are denoted as h_1 in Figure 5—and expressed the area of triangle ABC twice using the fact that it is two times the area of triangle BMC: $S = h_1 \cdot BM$; $S = \frac{BC \cdot AH}{2}$. Substituting AH by BM, Saul obtained that $h_1 = \frac{BC}{2}$. He was then surprised by the immediate implication of the last equation, namely, $\angle MBC = 30°$.

Saul:	The problem is about an inequality; I did not think that something would be found exactly.

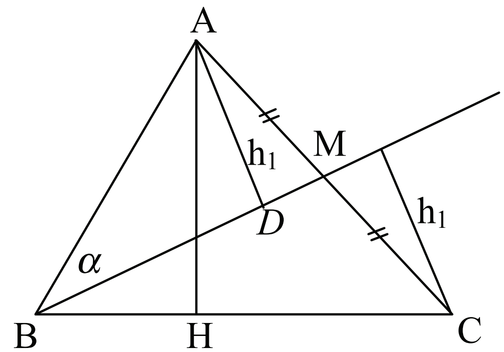

Figure 5. Saul solves the Angle Problem

Note that this assertion enlightens the role of encoding in Saul's actions. From here, Saul completed the solution exploiting the condition that AH is the greatest altitude and using the properties of the right angle triangle ABD. Namely, $AB \geq BC$, $AB \geq 2h_1$, $\frac{h_1}{AB} \leq \frac{1}{2}$, $\sin\alpha \leq \frac{1}{2}$, $\alpha \leq 30°$, and $\angle B \leq 60°$.

Afterwards, Saul explained retrospectively how he found the above solution:

Saul:	Do you want me to say what the point of this problem is? The point is that the two segments AH and BM are equal, and this directly reminded me of areas. AH is the altitude, it reminds me a formula of the area [of a triangle], and BM is a median, it reminds [me] that it divides the area of the triangle into two small triangles with equal areas, so I decided to think about areas . . . The next point was to discover that $h_1 = \frac{1}{2}BC$, and then I got that this [angle MBC] is 30. I decided to utilize the same idea here [in triangle ABD]. In fact, I saw that it is not the same situation, I could not have an exact expression, but I wanted to use somehow the fact that AH is the longest altitude, and from here, BC is the smallest.

Episode 5 Discussion

In this episode, Saul did not experience any conflict between effectiveness and elegance. Therefore, this is an example of a situation where the EEC is not observed. Our explanation is as follows. Unlike Mike and Alex, Saul systematically explored the given and only then drew an auxiliary construction. He also spent more time trying to invent a nontrigonometric solution than did Mike or Alex.

Indeed, Saul made essential progress after 2 minutes of heuristic search, whereas Mike and Alex considered switching a direction after 10–15 seconds of search without results. Apparently, Saul was trying to invent a solution online while his friends, especially Alex, relied more on memory retrieval. Saul's key auxiliary construction, triggered by the chain of associations (see Saul's retrospective explanations above), helped him to assemble the given into an equation, not into one triangle. Thus, Saul's solution was both algebra-laden and geometry-laden. The student was proud of this solution and was willing to explain it to the interviewer in great detail. The solution definitely fit his criteria of elegance because it exploited only elementary tools, which are included in every basic geometry curriculum. This special kind of parsimony was important to Saul because he put time and effort into a solution that would satisfy him aesthetically, even though he had well-polished algebraic methods in his arsenal, as will be evident from Episode 6. Another variation of Occam's principle of parsimony captures Saul's conception of elegance in problem solving: For the sake of elegance, one should use no more mathematical tools than the minimum needed.

Episode 6: Saul Solves the Bisector Problem

When reading the Bisector Problem, the following interaction took place:

Saul:	This is a famous problem! I read it in some book.
Interviewer:	Do you remember its solution?
Saul:	No, I don't [while speaking, Saul drew a picture]. It is possible to handle it by trigonometry . . . I think.
Interviewer:	Did you read a trigonometric solution in the book?
Saul:	No, it was another proof, geometric. But it is clear how to express everything as an equation, and then just to derive that $\alpha = \beta$.
Interviewer:	When you read a solution of a problem in a book, what do you try to memorize for use in the future?
Saul:	I learn new ideas, something that is worthwhile to remember. I remember that it was a difficult problem, but there were no new ideas in it, that's why I don't remember its solution.

Note that unlike Alex, Saul did not concentrate on retrieving the proof that he had read. Like Mike, Saul remembered that the problem was difficult. Apparently, this specific memory along with the fact that the problem was about angles triggered the following trigonometric solution.

Saul denoted the target angles as 2α and 2β, expressed the given that the two bisectors are equal as trigonometric equation by utilizing the theorem of sines, and stated his goal—to prove that $\alpha = \beta$ using trigonometry formulas. He achieved this goal confidently and quickly, but did not look happy with his solution.

Interviewer:	I see that something is bothering you in your solution.
Saul:	Of course, I don't like it; it is not nice to solve [geometry problems] by trigonometry.

As one can see, Saul solved the two problems by means of equations, but last time he used trigonometry extensively and articulated what we have called EEC as explicitly as he did it in his statement quoted in the prologue.

The interviewer asked Saul to find another solution to the problem, and the student tried to do so for about 35 minutes. He made five different attempts to solve the problem. In the first two, Saul considered the ideas from his previous trigonometric solution. In the other attempts, he tried to solve the problem arguing for a contradiction along with different auxiliary constructions. Some of them could have actually cracked the Bisector Problem. In fact, Saul articulated all the ideas needed to solve the problem, but could not combine them selectively in a completed solution path, and finally gave up.

Episode 6 Discussion

Saul decided not to take chances and to solve the Bisector Problem by means of trigonometry. He did it, and found himself encountering the conflict between effectiveness and elegance—the EEC. We think that the EEC was rooted in the conflict between two versions of the principle of parsimony, which were both important to the student. The first version was "for the sake of effectiveness, one should not make more efforts than the minimum needed," and the second one was, "for the sake of elegance, one should not use more mathematical tools than the minimum needed."

Additionally, Saul was more persistent than his peers in inventing aesthetically satisfying solutions. He was able to overcome the Law of Non-Deletion (see Episode 2) and to make more progress in solving the Bisector Problem geometrically than did his peers. Indeed, he advanced

beyond the attempts to recall the solution from the book (as Alex did) and the attempts to translate the algebraic solution at hand into geometry terms (as Mike did). It seems that considerations of elegance played both evaluative and generative roles in Saul's geometrical thinking.

Summary and Concluding Remarks

We described the following phenomenon: Gifted high school students intensively trained in solving Olympiad-style mathematics problems experience conflict between their conceptions of effectiveness and elegance. This conflict is referred to as the EEC. The students manifested different degrees of the EEC, apparently depending on the relative importance of the subprocess of retrieval in their geometry problem solving. As a rule, the more noticeable memory retrieval was, the more noticeable the EEC was. To use Sinclair's (2004) terms, the EEC plays evaluative or generative roles in the gifted students' problem solving.

We suggest that the didactical roots of the EEC are in the nature of the students' preparation for mathematics competitions. The students were trained to act effectively in the IMO, where all genuine solution methods are acceptable and the problems have to be solved under time constraints. Problem-solving schemes with high spread and applicability, like the use of trigonometric equations and advanced formulas of analytic geometry, are valuable for the students because they help to solve many classes of geometry problems by unified and readily retrieved approaches. Indeed, such universal methods are goal-directed and effective; that is, they help to save time and intellectual effort. On the other hand, the students appreciated the intellectual effort because they often faced unknown problems where the memorized problem-solving schemes cannot be retrieved and applied straightforwardly. Briefly speaking, the students appreciated the effective universal methods that often work, but also perceived these methods as "less elegant" because their use is in contrast to the idea of development of their creativity and original thinking.

The EEC also has epistemological roots. On the one hand, Harel (1998) argued that human beings have an intellectual need to be puzzled. Apparently, applying too powerful and universal tools to geometry problems, which are likely to have solutions by means of basic geometry tools, deprives the knowledgeable problem solvers of the pleasure to be puzzled and intellectually challenged. On the other hand, two different versions of Occam's principle of parsimony are involved in an epistemological explanation of the EEC. First, one should not make more intellectual effort than the minimum needed. Second, one should use no more tools than the minimum needed. As we have shown, the EEC was observed when the students were influenced by these two kinds of parsimony, and was not observed otherwise.

At this point, let us recall that appreciation of the latter kind of parsimony as a constituent of elegance came from studies about professional mathematicians and graduate students (e.g., Dreyfus & Eisenberg, 1986; Silver & Metzger, 1989). Our research extends previous research by documenting multiple influences of different kinds of parsimony on geometry problem solving of gifted high school students.

To conclude, we must remark on the potential importance of the presented findings. First, our findings indirectly support the hypothesis that appreciation of elegance in problem solving comes with particular problem-solving experiences; thus, students' appreciation can probably be developed by engaging in such experiences. Second, Dreyfus and Eisenberg (1986) pointed out that one of the major goals of mathematics teaching is to lead students, all students, to appreciate aesthetics of mathematical endeavors. To achieve this goal, we need to better understand complicated mechanisms by which gifted and expert problem solvers construct and call into play aesthetics considerations. We believe that our paper is a step toward this goal.

References

Baker, A. (2004). Simplicity. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy*, retrieved November 18, 2005, from http://plato.stanford.edu/archives/win2004/entries/simplicity

Berzsenyi, G. (1999). Providing opportunities through competitions. In L. J. Sheffield (Ed.), *Developing mathematically promising students* (pp. 185–190). Reston, VA: NCTM.

Clement, J. (2000). Analysis of clinical interviews: Foundation and model viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates.

Coxeter, H., & Greitzer, S. (1967). *Geometry revisited*. Washington, DC: Mathematical Association of America.

Davis, R. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Beckenham, England: Croom Helm.

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thoughts. *For the Learning of Mathematics*, 6(1), 2–10.

Dreyfus, T., & Eisenberg, T. (1996). On different facets of mathematical thinking. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 253–284). Mahwah, NJ: Lawrence Erlbaum Associates.

Erickson, K., & Simon, H. (1993). *Protocol analysis: Verbal reports as data* (Rev. ed.). Cambridge, MA: MIT Press.

Goldin, G. A. (2002) Affect, meta-affect, and mathematical belief structures. In G. C. Leder, E. Penkonen, & G. Torner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 59–72). Dordrecht, Netherlands: Kluwer.

Gorodetsky, M., & Klavir, R. (2003). What can we learn from how gifted/average pupils describe their process of problem solving? [Electronic version]. *Learning and Instruction*, *13*, 305–325.

Grassl, R., & Mingus, T. (1999). Nurturing the growth of young mathematicians through mathematics contests. In L. J. Sheffield (Ed.), *Developing mathematically promising students* (pp. 291–296). Reston, VA: NCTM.

Hadamard, J. (1945). *The mathematician's mind: The physiology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Harel, G. (1998). The dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly, 105*, 497–507.

Koichu, B. (2003). *Junior high school students' heuristic behaviors in mathematical problem solving*. Unpublished doctoral dissertation, Technion, Israel.

Koichu, B., Berman, A., & Moore, M. (2004). Promotion of heuristic literacy in a regular mathematics classroom. *For the Learning of Mathematics*, 24(1), 33–39.

Krutetskii, V. (1976). The psychology of mathematical abilities in schoolchildren (J. Teller, Trans.). Chicago: University of Chicago Press. (Original work published 1968)

McGregor, G. D., Jr. (2001). Gifted and talented youth as philosophers of the new millennium. *Journal of Secondary Gifted Education*, *12*, 157–161.

Nokelainen, P., Tirri, K., & Campbell, J. R. (2004). Cross-cultural predictors of mathematical talent and academic productivity. *High Ability Study, 15*, 229–242.

Poincaré, H. (1946). *The foundation of science* (G. B. Halsed, Trans.). Lancaster, PA: Science Press. (Original work published 1913)

Prasolov, V. V. (1995). *Zadachi po planimetrii* [Problems in plane geometry]. Moscow: NCNMO.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.

Silver, E., & Metzger, W. (1989). Aesthetic influences on expert problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 59–74). New York: Springer-Verlag.

Sinclair, N. (2001). The aesthetics *is* relevant. *For the Learning of Mathematics*, *21*(1), 25–33.

Sinclair, N. (2004). The role of the aesthetics in mathematical inquiry. *Mathematical Thinking and Learning*, *6*, 261–284.

Sternberg, R. J., & Davidson, J. E. (1983). Insight in the gifted. *Educational Psychologist, 18*, 51–57.

Von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *Journal of Education Thought, 25*, 91–103.

End Notes

[1] All students' names in this paper are pseudonyms.

[2] The data presented here were collected at the piloting stage of Koichu's (2003) doctoral study conducted at the Department of Education in the Technology and Science at the Technion–Israel Institute of Technology.